

ON THE BLOCK COUNTING PROCESS AND THE FIXATION LINE OF THE BOLTHAUSEN–SZNITMAN COALESCENT

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Abstract

The block counting process and the fixation line of the Bolthausen–Sznitman coalescent are analyzed. Spectral decompositions for their generators and transition probabilities are provided leading to explicit expressions for functionals such as hitting probabilities and absorption times. It is furthermore shown that the block counting process and the fixation line of the Bolthausen–Sznitman n -coalescent, properly scaled, converge in the Skorohod topology to the Mittag–Leffler process and Neveu’s continuous-state branching process respectively as the sample size n tends to infinity. Strong relations to Siegmund duality and to Mehler semigroups and self-decomposability are pointed out.

Keywords: block counting process; Bolthausen–Sznitman coalescent; fixation line; Mehler semigroup; Mittag–Leffler process; Neveu’s continuous-state branching process; self-decomposability; Siegmund duality; spectral decomposition

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1 Introduction

Exchangeable coalescents are Markovian processes $(\Pi_t)_{t \geq 0}$ with state space \mathcal{P} , the set of partitions of $\mathbb{N} := \{1, 2, \dots\}$. These processes have attracted the interest of several researchers, mainly in biology, mathematics and physics, during the last decades. The full family of exchangeable coalescents (with simultaneous multiple collisions) is a class of partition valued Markovian processes with a rich probabilistic structure and hence important for mathematical studies. Moreover, coalescents are useful in mathematical population genetics to model the ancestry of a sample of individuals or genes and therefore important for biological applications.

Exchangeable coalescents with multiple collisions but without simultaneous multiple collisions are characterized by a measure Λ on the unit interval $[0, 1]$ and therefore called Λ -coalescents. For further information on these processes we refer the reader to the independent works of Pitman [26] and Sagitov [30]. The most important coalescent is probably the Kingman coalescent [16], which allows only for binary mergers of ancestral lineages. In this case the measure Λ is the Dirac measure at 0.

For $t \geq 0$ let N_t denote the number of blocks of Π_t and let $N_t^{(n)}$ denote the number of blocks of $\Pi_t^{(n)}$, where $\Pi_t^{(n)}$ denotes the partition of Π_t restricted to a sample of size $n \in \mathbb{N}$. The processes $(N_t)_{t \geq 0}$ and $(N_t^{(n)})_{t \geq 0}$ are called the block counting processes of $(\Pi_t)_{t \geq 0}$ and $(\Pi_t^{(n)})_{t \geq 0}$ respectively.

Hénard [13] introduced the so-called fixation line $(L_t)_{t \geq 0}$ of a Λ -coalescent. Recently [10] the fixation line was extended to arbitrary exchangeable coalescents. One possible definition of the fixation line is based on the lookdown construction going back to Donnelly and Kurtz [5, 6]. For further information on the fixation line we refer the reader to [10] and [13]. In the following $(L_t^{(n)})_{t \geq 0}$ denotes the fixation line with initial state $L_0^{(n)} = n$.

In this article we focus on the Bolthausen–Sznitman coalescent [4], which is the particular Λ -coalescent with Λ being the uniform distribution on the unit interval. The generator $Q = (q_{ij})_{i,j \in \mathbb{N}}$ of the block counting process and the generator $\Gamma = (\gamma_{ij})_{i,j \in \mathbb{N}}$ of the fixation line

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of the Bolthausen–Sznitman coalescent have entries (see, for example, [22, Eq. (1.1)] and [13, p. 3015, Eq. (2.8) with $\alpha = 1$])

$$q_{ij} = \begin{cases} \frac{i}{(i-j)(i-j+1)} & \text{for } j < i \\ 1-i & \text{for } j = i, \\ 0 & \text{for } j > i. \end{cases} \quad \text{and} \quad \gamma_{ij} = \begin{cases} \frac{i}{(j-i)(j-i+1)} & \text{for } j > i, \\ -i & \text{for } j = i, \\ 0 & \text{for } j < i. \end{cases}$$

The block counting process and the corresponding generator Q have been studied intensively in the literature. In this article we focus on both processes $(N_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ with an emphasis on the fixation line $(L_t)_{t \geq 0}$, which has been studied less intensively so far. We furthermore stress the duality relation between both processes.

In Section 2 the processes $(N_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ are analyzed with an emphasis on spectral decompositions. These spectral decompositions lead to explicit expressions for several functionals of these processes such as hitting probabilities and absorption times.

Section 3 deals with the behavior of the block counting process $(N_t^{(n)})_{t \geq 0}$ and the fixation line $(L_t^{(n)})_{t \geq 0}$ as the sample size n tends to infinity. The main convergence result (Theorem 3.1) states that both processes, properly scaled, converge in the Skorohod sense as $n \rightarrow \infty$ to the Mittag–Leffler process and to Neveu’s continuous-state branching process respectively. The proofs provided in Section 4 rely on both analytic and probabilistic arguments which demonstrates the interplay between analysis and probability. A short appendix collects some results of independent interest used in the proofs.

2 Spectral decompositions and applications

Spectral decompositions are of fundamental interest since they lead to diagonal representations of the corresponding operators or matrices which simplify many mathematical calculations and numerical computations significantly. Explicit spectral decompositions for (the block counting process of) the Kingman coalescent and the Bolthausen–Sznitman coalescent are provided in [17] and [22]. We are interested in analog spectral decompositions for the fixation line. A spectral decomposition of the generator Γ of the fixation line of the Kingman coalescent is provided in the appendix (Lemma 5.2) for completeness. Our first result (Theorem 2.1) provides an explicit spectral decomposition for the generator Γ of the fixation line of the Bolthausen–Sznitman coalescent. In the following $s(i, j)$ and $S(i, j)$ denote the Stirling numbers of the first and second kind respectively.

Theorem 2.1 (Spectral decomposition of the generator of the fixation line)

The generator $\Gamma = (\gamma_{ij})_{i, j \in \mathbb{N}}$ of the fixation line $(L_t)_{t \geq 0}$ of the Bolthausen–Sznitman coalescent has spectral decomposition $\Gamma = RDL$, where $D = (d_{ij})_{i, j \in \mathbb{N}}$ is the diagonal matrix with entries $d_{ij} = -i$ for $i = j$ and $d_{ij} = 0$ for $i \neq j$, and $R = (r_{ij})_{i, j \in \mathbb{N}}$ and $L = (l_{ij})_{i, j \in \mathbb{N}}$ are upper right triangular matrices with entries

$$r_{ij} = \frac{i!}{j!}(-1)^{i+j}S(j, i) \quad \text{and} \quad l_{ij} = \frac{i!}{j!}(-1)^{i+j}s(j, i), \quad i, j \in \mathbb{N}. \quad (1)$$

The following corollaries demonstrate that spectral decompositions are useful to analyze the underlying processes.

Corollary 2.2 (Branching property/transition probabilities of the fixation line)

For the Bolthausen–Sznitman coalescent, the random variable $L_t^{(i)}$ has probability generating function (pgf)

$$\mathbb{E}(z^{L_t^{(i)}}) = (1 - (1 - z)e^{-t})^i, \quad |z| < 1, t \geq 0, i \in \mathbb{N}. \quad (2)$$

Thus, $(L_t)_{t \geq 0}$ is a Markovian continuous-time branching process with state space \mathbb{N} and offspring distribution $p_k = 1/(k(k-1))$, $k \in \{2, 3, \dots\}$ having infinite mean. Moreover, the transition probabilities $p_{ij}(t) := \mathbb{P}(L_t = j \mid L_0 = i) = \mathbb{P}(L_t^{(i)} = j)$ are given by

$$p_{ij}(t) = (-1)^{i+j} \frac{i!}{j!} \sum_{k=i}^j S(k, i) e^{-tk} s(j, k) = (-1)^j \sum_{k=1}^i (-1)^k \binom{i}{k} \binom{k}{j} e^{-tk}, \quad i, j \in \mathbb{N}. \quad (3)$$

Remarks.

1. For $i = 1$ it follows that $L_t = L_t^{(1)}$ has pgf $\mathbb{E}(z^{L_t}) = 1 - (1 - z)^\alpha = -\sum_{j=1}^{\infty} \binom{\alpha}{j} (-z)^j$ and distribution

$$\mathbb{P}(L_t = j) = p_{1j}(t) = (-1)^{j+1} \binom{\alpha}{j} = \frac{\alpha \Gamma(j - \alpha)}{\Gamma(1 - \alpha) \Gamma(j + 1)}, \quad j \in \mathbb{N}, \quad (4)$$

where $\alpha := e^{-t}$. Note that $\mathbb{P}(L_t = j) \sim \alpha / (\Gamma(1 - \alpha) j^{\alpha+1})$ as $j \rightarrow \infty$ and that L_t has a Pareto like tail $\mathbb{P}(L_t \geq j) = \Gamma(j - \alpha) / (\Gamma(1 - \alpha) \Gamma(j)) \sim 1 / (\Gamma(1 - \alpha) j^\alpha)$ as $j \rightarrow \infty$. Thus, $\mathbb{E}(L_t^q) = \sum_{j=1}^{\infty} j^q \mathbb{P}(L_t = j) < \infty$ if and only if $q < \alpha$. Particular reciprocal factorial moments of L_t are known explicitly. For example,

$$\mathbb{E}\left(\frac{1}{(L_t + 1)(L_t + 2) \cdots (L_t + k)}\right) = \frac{\alpha}{\Gamma(1 - \alpha)} \sum_{j=1}^{\infty} \frac{\Gamma(j - \alpha)}{\Gamma(j + k + 1)} = \frac{\alpha}{k! (\alpha + k)}, \quad k \in \mathbb{N}.$$

The distribution (4) and similar distributions occur in [14, p. 9], [15, p. 225] and [27, p. 70, Eq. (3.38)].

2. The pgf $f(s) := \sum_{k=2}^{\infty} p_k s^k = s + (1 - s) \log(1 - s)$ of the offspring distribution satisfies

$$\int_{(1-\varepsilon, 1)} \frac{\lambda(ds)}{f(s) - s} = \int_{(1-\varepsilon, 1)} \frac{\lambda(ds)}{(1 - s) \log(1 - s)} = \int_{(0, \varepsilon)} \frac{\lambda(dx)}{x \log x} = -\infty$$

for all $\varepsilon \in (0, 1)$, where λ denotes Lebesgue measure on $(0, 1)$. This implies (Harris [12, p. 107]) that the fixation line $(L_t)_{t \geq 0}$ does not explode, in agreement (see [10]) with the fact that the Bolthausen–Sznitman coalescent stays infinite.

As a second application we study the probability $h(i, j) = \mathbb{P}(L_t^{(i)} = j \text{ for some } t \geq 0)$ that the fixation line hits state $j \in \mathbb{N}$ started from state $i \in \mathbb{N}$.

Corollary 2.3 (Hitting probabilities) *The hitting probabilities $h(i, j)$ have horizontal generating function*

$$\sum_{j=i}^{\infty} h(i, j) z^{j-1} = \frac{z^i}{(1 - z)(-\log(1 - z))}, \quad i \in \mathbb{N}, |z| < 1. \quad (5)$$

In particular $h(i, j) = h(1, j - i + 1)$ depends on i and j only via $j - i$. Moreover, for all $i \in \mathbb{N}$,

$$h(i, j) = \frac{1}{\log j} - \frac{\gamma}{\log^2 j} + O\left(\frac{1}{\log^3 j}\right), \quad j \rightarrow \infty, \quad (6)$$

where $\gamma := -\Gamma'(1) \approx 0.577216$ denotes the Euler–Mascheroni constant. The hitting probability $h(i, j)$ can be computed via

$$h(i, j) = \sum_{k=1}^{j-i} \mathbb{P}(\eta_1 + \cdots + \eta_k = j - i), \quad 1 \leq i < j, \quad (7)$$

where η_1, η_2, \dots are iid random variables with distribution $\mathbb{P}(\eta_1 = n) := u_n := 1/(n(n+1))$, $n \in \mathbb{N}$. The hitting probabilities can be also expressed in terms of the Stirling numbers $s(.,.)$ and $S(.,.)$ of the first and second kind as

$$h(i, j) = (-1)^{i+j} \frac{i!}{(j-1)!} \sum_{k=i}^j \frac{s(j, k) S(k, i)}{k} \quad (8)$$

$$= (-1)^{j-i} \frac{1}{(j-i)!} \sum_{k=1}^{j-i+1} \frac{s(j-i+1, k)}{k}, \quad 1 \leq i \leq j. \quad (9)$$

Moreover, $h(i, j)$ has representations

$$h(i, j) = \frac{1}{(j-i)!} \int_0^1 \frac{\Gamma(j-i+x)}{\Gamma(x)} dx = \frac{1}{(j-i)!} \sum_{k=0}^{j-i} \frac{|s(j-i, k)|}{k+1}, \quad 1 \leq i \leq j. \quad (10)$$

Remark. Concrete values of the hitting probabilities $h(i, j)$ for $i = 1$ and $j \in \{1, \dots, 7\}$ are provided in the remark after the proof of Corollary 2.3.

We now turn to the block counting process $(N_t^{(n)})_{t \geq 0}$ of the Bolthausen–Sznitman n -coalescent. For $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ let $\tau_{ni} := \inf\{t > 0 : N_t^{(n)} \leq i\}$ denote the first time the block counting process $(N_t^{(n)})_{t \geq 0}$ jumps to a state smaller than or equal to i . Note that $\tau_n := \tau_{n1}$ is the absorption time of $N^{(n)}$.

Corollary 2.4 (Distribution function and asymptotics of τ_{ni}) *For all $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, τ_{ni} has distribution function*

$$\mathbb{P}(\tau_{ni} \leq t) = \sum_{j=1}^i (-1)^{n+j} \binom{i}{j} \binom{je^{-t} - 1}{n-1}, \quad t \in (0, \infty). \quad (11)$$

In particular, for every $i \in \mathbb{N}$, $\tau_{ni} - \log \log n \rightarrow \min(G_1, \dots, G_i)$ in distribution as $n \rightarrow \infty$, where G_1, G_2, \dots are independent standard Gumbel distributed random variables.

Remark. Note that $\min(G_1, \dots, G_i)$ has distribution function $F_i(x) := 1 - (1 - F(x))^i$, where $F(x) := e^{-e^{-x}}$, $x \in \mathbb{R}$. For $i = 1$ we recover the well known convergence result (see Goldschmidt and Martin [11, Proposition 3.4], Freund and Möhle [9, Corollary 1.2] or Hénard [13, Theorem 3.9]) that the scaled absorption time $\tau_n - \log \log n$ is asymptotically standard Gumbel distributed.

The fact that the distribution function (11) of τ_{ni} is known explicitly can be further exploited. For example, the following Edgeworth expansion holds.

Corollary 2.5 (Edgeworth expansion) *For every $i \in \mathbb{N}$ and $x \in \mathbb{R}$ the following Edgeworth expansion of order $K \in \mathbb{N}_0$ holds.*

$$\mathbb{P}(\tau_{ni} - \log \log n \leq x) = \sum_{k=0}^K c_k d_{ki}(x) \frac{e^{-kx}}{\log^k n} + O\left(\frac{1}{\log^{K+1} n}\right), \quad n \rightarrow \infty, \quad (12)$$

where c_0, c_1, \dots are the coefficients in the series expansion $1/\Gamma(1-x) = \sum_{k=0}^{\infty} c_k x^k$, $|x| < 1$, and

$$d_{ki}(x) := \left(e^x \frac{d}{dx}\right)^k F_i(x) = \sum_{j=1}^i (F(x))^j (-1)^{j-1} \binom{i}{j} j^k, \quad k \in \mathbb{N}_0, i \in \mathbb{N}, x \in \mathbb{R}, \quad (13)$$

with F_i and F as defined in the previous remark. Alternatively, $d_{0i}(x) = F_i(x)$ and

$$d_{ki}(x) = \sum_{j=1}^k S(k, j) (-1)^{j-1} (i)_j (F(x))^j (1 - F(x))^{i-j}, \quad k, i \in \mathbb{N}, x \in \mathbb{R}, \quad (14)$$

where the $S(k, j)$ are the Stirling numbers of the second kind and $(i)_j := i(i-1) \cdots (i-j+1)$.

Remarks.

1. The coefficients c_k , $k \in \mathbb{N}_0$, are related to the moments of the Gumbel distribution (see Lemma 4.1). The concrete values c_k for $k \leq 3$ are provided in the remark after the proof of Lemma 4.1.
2. For $K = 1$ Corollary 2.5 reads $\mathbb{P}(\tau_{ni} - \log \log n \leq x) = F_i(x) - \gamma F'_i(x)/\log n + O(1/\log^2 n)$. In particular, for every $x \in \mathbb{R}$, $\mathbb{P}(\tau_{ni} - \log \log n \leq x) - F_i(x) \sim -\gamma F'_i(x)/\log n$ as $n \rightarrow \infty$. Thus, the speed of the convergence of $\tau_{ni} - \log \log n$ to G_i is of order $1/\log n$.

3 Asymptotics

We are interested in the behavior of the block counting process $(N_t^{(n)})_{t \geq 0}$ and the fixation line $(L_t^{(n)})_{t \geq 0}$ of the Bolthausen–Sznitman n -coalescent as the sample size n tends to infinity. In order to state the main convergence result (see Theorem 3.1 below) let us recall some properties of the Mittag–Leffler process $X = (X_t)_{t \geq 0}$ and Neveu’s [23] continuous-state branching process $Y = (Y_t)_{t \geq 0}$.

The Mittag–Leffler process X is a Markovian process in continuous time with state space $E := [0, \infty)$. The name of this process comes from the fact that for every $t \geq 0$ the marginal random variable X_t is Mittag–Leffler distributed with parameter e^{-t} . Note that X_t has moments $\mathbb{E}(X_t^m) = \Gamma(1+m)/\Gamma(1+me^{-t})$, $m \in [0, \infty)$. The semigroup $(T_t^X)_{t \geq 0}$ of the Mittag–Leffler process X is given by

$$T_t^X f(x) = \mathbb{E}(f(xe^{-t} X_t)), \quad t, x \geq 0, f \in B(E), \quad (15)$$

where $B(E)$ denotes the set of all bounded measurable functions $f : E \rightarrow \mathbb{R}$. Some further information on the process X can be found in [2] and [21].

Neveu’s [23] continuous-state branching process Y is as well a Markovian process in continuous time with state space E . For every $t \geq 0$ the marginal random variable Y_t is α -stable with Laplace transform $\mathbb{E}(e^{-\lambda Y_t}) = e^{-\lambda^\alpha}$, $\lambda \geq 0$, where $\alpha := e^{-t}$. The semigroup $(T_t^Y)_{t \geq 0}$ of Neveu’s continuous-state branching process Y is given by

$$T_t^Y g(y) = \mathbb{E}(g(ye^t Y_t)), \quad t, y \geq 0, g \in B(E). \quad (16)$$

Note that (see, for example, [21]) the Mittag–Leffler process X is Siegmund dual to Neveu’s continuous state branching process Y , i.e. $\mathbb{P}(X_t \leq y | X_0 = x) = \mathbb{P}(Y_t \geq x | Y_0 = y)$ for all $t, x, y \geq 0$.

Define the scaled block counting process $X^{(n)} = (X_t^{(n)})_{t \geq 0}$ and the scaled fixation line $Y^{(n)} := (Y_t^{(n)})_{t \geq 0}$ of the Bolthausen–Sznitman n -coalescent via

$$X_t^{(n)} := \frac{N_t^{(n)}}{ne^{-t}} \quad \text{and} \quad Y_t^{(n)} := \frac{L_t^{(n)}}{ne^t}, \quad t \geq 0, n \in \mathbb{N}. \quad (17)$$

Note that, for $n \geq 2$, the processes $X^{(n)}$ and $Y^{(n)}$ are time-inhomogeneous because of the time-dependent scalings ne^{-t} and ne^t . We are now able to state the main convergence result. The proof of the following theorem is provided in Section 4.

Theorem 3.1 (Asymptotics of the block counting process and the fixation line)

For the Bolthausen–Sznitman coalescent the following two assertions hold.

- a) *As $n \rightarrow \infty$ the scaled block counting process $X^{(n)}$, defined in (17), converges in $D_E[0, \infty)$ to the Mittag–Leffler process $X = (X_t)_{t \geq 0}$.*
- b) *As $n \rightarrow \infty$ the scaled fixation line $Y^{(n)}$, defined in (17), converges in $D_E[0, \infty)$ to Neveu’s continuous-state branching process $Y = (Y_t)_{t \geq 0}$.*

Theorem 3.1 demonstrates the intimate relation between the Bolthausen–Sznitman coalescent, the Mittag–Leffler process and Neveu’s continuous state branching process. We refer the reader to Bertoin and Le Gall [3] for further insights concerning these relations.

Theorem 3.1 a) is known from the literature [21, Theorem 1.1] and provided here for completeness. Our proof of Theorem 3.1 a) is significantly shorter than the proof provided in [21] and gives further insights into the structure of the scaled block counting process $X^{(n)}$.

Part b) of Theorem 3.1 is likely to be known from branching process theory, however the authors have not been able to trace this result in the literature. Note that the offspring distribution of the branching process $(L_t^{(n)})_{t \geq 0}$ has pgf $f(s) = s + (1-s)\log(1-s)$ and, hence, infinite mean. For related convergence results for the critical case when the offspring distribution has mean 1 we refer the reader to Sagitov [29] and the references therein. Note

that in Theorem 2.1 of [29] the space-scaling is n and an additional time-scaling occurs. Theorem 3.1 b) may be viewed as a kind of boundary case of Theorem 2.1 of [29] for $\alpha \rightarrow 1$. Similar convergence results for sequences of discrete-time branching processes can be traced back to Lamperti [18, 19].

In summary the following commutative diagram holds.

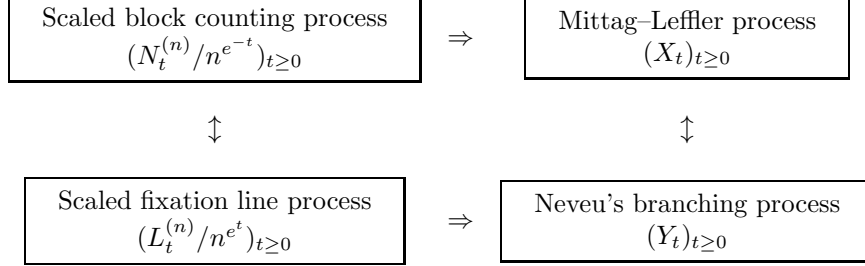


Figure 1: Commutative diagram for the block counting process $(N_t^{(n)})_{t \geq 0}$ and the fixation line $(L_t^{(n)})_{t \geq 0}$ of the Bolthausen–Sznitman coalescent. The right-arrows ‘ \Rightarrow ’ stand for ‘convergence in $D_E[0, \infty)$ as $n \rightarrow \infty$ ’. The vertical updown-arrows ‘ \updownarrow ’ stand for ‘duality’, on the left hand side the duality of the block counting process $(N_t)_{t \geq 0}$ and the fixation line $(L_t)_{t \geq 0}$ with respect to the Siegmund duality kernel $H : \mathbb{N}^2 \rightarrow \{0, 1\}$ defined via $H(i, j) := 1$ for $i \leq j$ and $H(i, j) := 0$ otherwise, on the right hand side the duality of $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ with respect to the Siegmund duality kernel $H : [0, \infty)^2 \rightarrow \{0, 1\}$ defined via $H(x, y) := 1$ for $x \leq y$ and $H(x, y) := 0$ otherwise.

We finally point out that Theorem 3.1 is strongly related to Mehler semigroups, to self-decomposability and to the Gumbel distribution. Clearly, Theorem 3.1 can be stated logarithmically as follows. The process $(\log N_t^{(n)} - e^{-t} \log n)_{t \geq 0}$ converges in $D_{\mathbb{R}}[0, \infty)$ to $\tilde{X} := (\tilde{X}_t)_{t \geq 0} := (\log X_t)_{t \geq 0}$ and the process $(\log L_t^{(n)} - e^t \log n)_{t \geq 0}$ converges in $D_{\mathbb{R}}[0, \infty)$ to $\tilde{Y} := (\tilde{Y}_t)_{t \geq 0} := (\log Y_t)_{t \geq 0}$ as $n \rightarrow \infty$. Note that the semigroup $(T_t^{\tilde{X}})_{t \geq 0}$ of \tilde{X} is given by

$$T_t^{\tilde{X}} f(x) = \mathbb{E}(f(xe^{-t} + \tilde{X}_t)), \quad t \geq 0, f \in B(\mathbb{R}), x \in \mathbb{R}, \quad (18)$$

whereas the semigroup $(T_t^{\tilde{Y}})_{t \geq 0}$ of \tilde{Y} is given by

$$T_t^{\tilde{Y}} g(y) = \mathbb{E}(g(ye^t + \tilde{Y}_t)), \quad t \geq 0, g \in B(\mathbb{R}), y \in \mathbb{R}. \quad (19)$$

Semigroups of this form belong to the class of so called Mehler semigroups. Note that (18) and (19) define the semigroups of \tilde{X} and \tilde{Y} completely, since for every $t \geq 0$ the distributions of the marginals $\tilde{X}_t = \log X_t$ and $\tilde{Y}_t = \log Y_t$ can be characterized as follows. Let E be standard exponentially distributed and independent of X and Y . Note that $G := -\log E$ is standard Gumbel distributed. From $E \stackrel{d}{=} (E/Y_t)e^{-t}$ (see, for example, [31]) we conclude by an application of the transformation $x \mapsto -\log x$ that the distribution of \tilde{Y}_t is characterized via the self-decomposable distributional equation

$$G \stackrel{d}{=} e^{-t}G + e^{-t}\tilde{Y}_t.$$

Thus, \tilde{Y}_t has characteristic function $u \mapsto \Gamma(1 - iue^t)/\Gamma(1 - iu)$, $u \in \mathbb{R}$, and cumulants $\kappa_j(\tilde{Y}_t) = (e^{jt} - 1)\kappa_j(G)$, $j \in \mathbb{N}$, $t \geq 0$, where $\kappa_j(G)$ are the cumulants of the Gumbel distribution, i.e. $\kappa_1(G) = \gamma$ (Euler–Mascheroni constant) and $\kappa_j(G) = (-1)^j \Psi^{(j-1)}(1) = (j-1)!\zeta(j)$ for $j \in \mathbb{N} \setminus \{1\}$, where Ψ and ζ denote the digamma function (logarithmic derivative of the gamma function) and the Riemann zeta function respectively.

Similarly, the distribution of \tilde{X}_t is characterized via the self-decomposable distributional equation

$$S \stackrel{d}{=} e^{-t}S + \tilde{X}_t,$$

where $S := -G$. Therefore, \tilde{X}_t has characteristic function $u \mapsto \Gamma(1 + iu)/\Gamma(1 + iue^{-t})$, $u \in \mathbb{R}$, and cumulants $\kappa_j(\tilde{X}_t) = (-1)^j(1 - e^{-jt})\kappa_j(G)$, $j \in \mathbb{N}$, $t \geq 0$.

4 Proofs

Proof. (of Theorem 2.1) Two proofs are provided. The first proof is self-contained and based on generating functions. The second proof uses duality and the spectral decomposition [22, Theorem 1.1] of the generator of the block counting process.

Proof 1. (via generating functions)

The proof is similar to that of Theorem 1.1 of [22]. Let $D = (d_{ij})_{i,j \in \mathbb{N}}$ be the diagonal matrix with entries $d_{ii} := -\gamma_i = \gamma_{ii}$, $i \in \mathbb{N}$. Furthermore, let $R = (r_{ij})_{i,j \in \mathbb{N}}$ be the upper right triangular matrix with entries defined for each $j \in \mathbb{N}$ recursively via $r_{jj} := 1$ and

$$r_{ij} := \frac{1}{\gamma_i - \gamma_j} \sum_{k=i+1}^j \gamma_{ik} r_{kj}, \quad i \in \{j-1, j-2, \dots, 1\}. \quad (20)$$

Since $\gamma_{ii} = -\gamma_i$, $i \in \mathbb{N}$, we conclude that $r_{ij} \gamma_{jj} = \sum_{k=i}^j \gamma_{ik} r_{kj}$. Thus, the entries of R are defined such that $RD = \Gamma R$. Define $L := R^{-1}$. Then, the spectral decomposition $\Gamma = RDL$ holds. Moreover, $DL = L\Gamma$ and, hence, $\gamma_{ii} l_{ij} = \sum_{k=i}^j l_{ik} \gamma_{kj}$, $i, j \in \mathbb{N}$. Since $\gamma_{ii} = -\gamma_i$, $i \in \mathbb{N}$, we obtain for each $i \in \mathbb{N}$ the recursion $l_{ii} = 1$ and

$$l_{ij} = \frac{1}{\gamma_j - \gamma_i} \sum_{k=i}^{j-1} l_{ik} \gamma_{kj}, \quad j \in \{i+1, i+2, \dots\}. \quad (21)$$

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc. For $i \in \mathbb{N}$ define the generating function $l_i : U \rightarrow \mathbb{C}$ via $l_i(z) := \sum_{j=i}^{\infty} l_{ij} z^j$, $z \in U$, and consider the modified function $f_i : U \rightarrow \mathbb{C}$ defined via $f_i(z) := \sum_{j=i}^{\infty} (j-i) l_{ij} z^j$, $z \in U$. We have

$$f_i(z) = \sum_{j=i}^{\infty} j l_{ij} z^j - i \sum_{j=i}^{\infty} l_{ij} z^j = z l'_i(z) - i l_i(z).$$

On the other hand, by the recursion (21), we obtain the factorization

$$\begin{aligned} f_i(z) &= \sum_{j=i+1}^{\infty} (j-i) l_{ij} z^j = \sum_{j=i+1}^{\infty} \sum_{k=i}^{j-1} l_{ik} \gamma_{kj} z^j \\ &= \sum_{k=i}^{\infty} l_{ik} \sum_{j=k+1}^{\infty} \gamma_{kj} z^j = \sum_{k=i}^{\infty} k l_{ik} z^k \sum_{j=k+1}^{\infty} \frac{z^{j-k}}{(j-k)(j-k+1)} \\ &= \sum_{k=i}^{\infty} k l_{ik} z^k \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} = z l'_i(z) a(z), \end{aligned}$$

where the auxiliary function $a : U \rightarrow \mathbb{C}$ is defined via $a(z) := \sum_{n=1}^{\infty} z^n / (n(n+1)) = 1 - (1-z)(-\log(1-z))/z$, $z \in U$. Thus, l_i satisfies the differential equation $z l'_i(z) a(z) = z l'_i(z) - i l_i(z)$ or, equivalently,

$$l'_i(z) = \frac{i l_i(z)}{(1-a(z))z} = \frac{i l_i(z)}{(1-z)(-\log(1-z))}.$$

The solution of this homogeneous differential equation with initial conditions $l_i(0) = \dots = l_i^{(i-1)}(0) = 0$ and $l_i^{(i)}(0) = i!$ is $l_i(z) = (-\log(1-z))^i$, $i \in \mathbb{N}$, $z \in U$. Here $l_i^{(j)}$ denotes the j th derivative of l_i . For $f(z) = \sum_{j=0}^{\infty} a_j z^j$ let $[z^j]f(z) := a_j$ denote the coefficient in front of z^j in the series expansion of f . By [1, p. 824], $l_i(z) = (-\log(1-z))^i = i! \sum_{j=i}^{\infty} |s(j, i)| z^j / j!$ and, hence,

$$l_{ij} = [z^j] l_i(z) = \frac{i!}{j!} |s(j, i)| = \frac{i!}{j!} (-1)^{i+j} s(j, i),$$

which is the second formula in (1). Let us now turn to the inverse $R = L^{-1}$ of L . We have $L(z, z^2, \dots)^\top = (l_1(z), l_2(z), \dots)^\top$. Multiplying from the left with R it follows that

$(z, z^2, \dots)^\top = R(l_1(z), l_2(z), \dots)^\top$. Thus, $z^i = \sum_{j=i}^\infty r_{ij} l_j(z) = \sum_{j=i}^\infty r_{ij} (-\log(1-z))^j$. Replacing z by $1 - e^{-z}$ leads to $(1 - e^{-z})^i = \sum_{j=i}^\infty r_{ij} z^j =: r_i(z)$, $i \in \mathbb{N}$, $z \in U$. The calculations between Eq. (2.9) and Eq. (2.10) in [22] show that r_i has expansion

$$r_i(z) = (1 - e^{-z})^i = \sum_{j=0}^\infty (-1)^{i+j} \frac{i!}{j!} S(j, i) z^j,$$

which yields the formula in (1) for the coefficient $r_{ij} = [z^j]r_i(z)$ in front of z^j . \square

Proof 2. (via duality)

The duality kernel H can be interpreted as a non-singular matrix $H = (h_{ij})_{i,j \in \mathbb{N}}$ with entries $h_{ij} = 1$ for $j \geq i$ and $h_{ij} = 0$ for $j < i$. The entries of its inverse $H^{-1} =: (g_{ij})_{i,j \in \mathbb{N}}$ are given by $g_{ij} = \delta_{i,j} - \delta_{i+1,j}$. It is known [22] that the generator matrix Q of the block counting process has spectral decomposition $Q = \tilde{R} \tilde{D} \tilde{L}$, where the matrices $\tilde{R} = (\tilde{r}_{ij})_{i,j \in \mathbb{N}}$, $\tilde{D} = (\tilde{d}_{ij})_{i,j \in \mathbb{N}}$ and $\tilde{L} = (\tilde{l}_{ij})_{i,j \in \mathbb{N}}$ are given by $\tilde{r}_{ij} = ((j-1)!/(i-1)!)|s(i, j)|$, $\tilde{d}_{ij} = (i-1)\delta_{i,j}$ and $\tilde{l}_{ij} = (-1)^{i+j}((j-1)!/(i-1)!S(i, j))$ respectively. The entries of $D = (d_{ij})_{i,j \in \mathbb{N}}$ can be read from the diagonal of Γ and are therefore given by $d_{ij} = i\delta_{i,j}$. Define the matrices $A = (a_{ij})_{i,j \in \mathbb{N}}$ and $B = (b_{ij})_{i,j \in \mathbb{N}}$ by $a_{ij} = \delta_{i+1,j}$ and $b_{ij} = \delta_{i-1,j}$. Clearly $\tilde{D} = BDA$. This together with the duality relation $H\Gamma^\top = QH$ and the spectral decomposition of the block counting process $Q = \tilde{R} \tilde{D} \tilde{L}$ yields

$$\Gamma^\top = H^{-1} \tilde{R} \tilde{D} \tilde{L} H = (-H^{-1} \tilde{R} B) D (-A \tilde{L} H).$$

Hence $\Gamma = RDL$ with $R := (-A \tilde{L} H)^\top$ and $L := (-H^{-1} \tilde{R} B)^\top$. It remains to calculate the entries of R and L . Using the recursion $S(i+1, j) = jS(i, j) + S(i, j-1)$ we obtain

$$\begin{aligned} r_{ji} &= (-A \tilde{L} H)_{ij} = -(\tilde{L} H)_{i+1,j} = -\sum_{k=1}^j \tilde{l}_{i+1,k} = \sum_{k=1}^j (-1)^{i+k} \frac{(k-1)!}{i!} S(i+1, k) \\ &= \sum_{k=1}^j (-1)^{i+k} \frac{k!}{i!} S(i, k) + \sum_{k=1}^j (-1)^{i+k} \frac{(k-1)!}{i!} S(i, k-1) \\ &= \sum_{k=1}^j (-1)^{i+k} \frac{k!}{i!} S(i, k) - \sum_{k=0}^{j-1} (-1)^{i+k} \frac{k!}{i!} S(i, k) = (-1)^{i+j} \frac{j!}{i!} S(i, j). \end{aligned}$$

Using the recursion $|s(i+1, j+1)| = |s(i, j)| + i|s(i, j+1)|$ we get

$$\begin{aligned} l_{ji} &= (-H^{-1} \tilde{R} B)_{ij} = -(H^{-1} \tilde{R})_{i,j+1} = \tilde{r}_{i+1,j+1} - \tilde{r}_{i,j+1} \\ &= \frac{j!}{i!} |s(i+1, j+1)| - \frac{j!}{(i-1)!} |s(i, j+1)| = \frac{j!}{i!} |s(i, j)|. \end{aligned}$$

\square

Proof. (of Corollary 2.2) By Theorem 2.1, $\Gamma = RDL$, where R and $L = R^{-1}$ have entries (1). Hence, the transition matrix $P(t) = e^{t\Gamma}$ has spectral decomposition $P(t) = e^{tRDL} = Re^{tD}L$. Thus, $p_{ij}(t) = \mathbb{P}(L_t = j \mid L_0 = i) = (Re^{tD}L)_{ij} = \sum_{k=i}^j r_{ik} e^{-\gamma_k t} l_{kj}$. The first formula in (3) for $p_{ij}(t)$ follows from $\gamma_k = k$ and from (1). Recall that $\alpha := e^{-t}$. Conditional on $L_0 = i$ the random variable L_t has probability generating function

$$\begin{aligned} \mathbb{E}(z^{L_t} \mid L_0 = i) &= \sum_{j=i}^\infty z^j p_{ij}(t) = \sum_{j=i}^\infty z^j (-1)^{i+j} \frac{i!}{j!} \sum_{k=i}^j S(k, i) \alpha^k s(j, k) \\ &= (-1)^i i! \sum_{k=i}^\infty S(k, i) \alpha^k \sum_{j=k}^\infty \frac{(-z)^j}{j!} s(j, k) \\ &= (-1)^i i! \sum_{k=i}^\infty S(k, i) \alpha^k \frac{(\log(1-z))^k}{k!} \\ &= (-1)^i (e^{\alpha \log(1-z)} - 1)^i = (1 - (1-z)^\alpha)^i, \quad |z| < 1, t \geq 0, i \in \mathbb{N}. \end{aligned}$$

Expansion leads to

$$\begin{aligned}\mathbb{E}(z^{L_t} | L_0 = i) &= \sum_{k=0}^i \binom{i}{k} (-1)^k (1-z)^{\alpha k} = \sum_{k=0}^i \binom{i}{k} (-1)^k \sum_{j=0}^{\infty} \binom{\alpha k}{j} (-z)^j \\ &= \sum_{j=0}^{\infty} (-z)^j \sum_{k=0}^i (-1)^k \binom{i}{k} \binom{\alpha k}{j}.\end{aligned}$$

The coefficient in front of z^j in this expansion yields the second formula for $p_{ij}(t)$. \square

Proof. (of Corollary 2.3) The hitting probability $h(i, j)$ is related to the entry $g(i, j) := \int_0^\infty \mathbb{P}(L_t^{(i)} = j) dt$ of the Green matrix via $h(i, j) = \gamma_j g(i, j) = j g(i, j)$ (see, for example, Norris [24, p. 146]). Thus, for all $i \in \mathbb{N}$ and $|z| < 1$,

$$h_i(z) := \sum_{j=i}^{\infty} h(i, j) z^{j-1} = \int_0^\infty \sum_{j=i}^{\infty} j \mathbb{P}(L_t^{(i)} = j) z^{j-1} dt = \int_0^\infty \frac{d}{dz} \sum_{j=i}^{\infty} \mathbb{P}(L_t^{(i)} = j) z^j dt.$$

Plugging in the formula (2) for the pgf of $L_t^{(i)}$ it follows that

$$h_i(z) = \int_0^\infty \frac{d}{dz} (1 - (1-z)^{e^{-t}})^i dt = \int_0^\infty i (1 - (1-z)^{e^{-t}})^{i-1} e^{-t} (1-z)^{e^{-t}-1} dt.$$

Substituting $x := e^{-t}$ and noting that $dt/dx = -1/x$ leads to $h_i(z) = (1-z)^{-1} \int_0^1 i (1 - (1-z)^x)^{i-1} (1-z)^x dx$. Substituting further $y := 1 - (1-z)^x$ and noting that $dx/dy = 1/((1-y)(-\log(1-z)))$ we obtain

$$h_i(z) = \frac{1}{(1-z)(-\log(1-z))} \int_0^z i y^{i-1} dy = \frac{z^i}{(1-z)(-\log(1-z))}, \quad i \in \mathbb{N}, |z| < 1.$$

In particular, $h(i, j) = h(1, j-i+1)$. The asymptotic expansion (6) follows from Panholzer [25, Eq. (19)]. Formula (7) is obtained as follows. Let $(J_k)_{k \in \mathbb{N}_0}$ denote the jump chain of the fixation line $(L_t)_{t \geq 0}$. Given this chain is in state i it jumps to state $i+j$ with probability $\gamma_{i,i+j}/\gamma_i = 1/(j(j+1)) =: u_j$, $j \in \mathbb{N}$. From this property it is easily seen that the jump chain has independent increments, i.e. $J_0 = 1$, $J_1 = 1 + \eta_1$, $J_2 = 1 + \eta_1 + \eta_2$ and so on, where η_1, η_2, \dots are iid random variables with distribution $\mathbb{P}(\eta_1 = j) = u_j$, $j \in \mathbb{N}$. For $1 \leq i < j$ it follows that $h(i, j) = h(1, j-i+1) = \sum_{k=1}^{j-i} \mathbb{P}(J_k = j-i+1) = \sum_{k=1}^{j-i} \mathbb{P}(\eta_1 + \dots + \eta_k = j-i)$. Formula (8) for $h(i, j)$ follows from $h(i, j) = j g(i, j) = j \int_0^\infty \mathbb{P}(L_t^{(i)} = j) dt$ and

$$\begin{aligned}\int_0^\infty \mathbb{P}(L_t^{(i)} = j) dt &= \int_0^\infty (-1)^{i+j} \frac{i!}{j!} \sum_{k=i}^j S(k, i) e^{-tk} s(j, k) dt \\ &= (-1)^{i+j} \frac{i!}{j!} \sum_{k=i}^j \frac{S(k, i) s(j, k)}{k}.\end{aligned}$$

Eq. (9) follows from $h(i, j) = h(1, j-i+1)$ and $S(k, 1) = 1$ for all $k \in \mathbb{N}$. Moreover, for $i = 1$ we have $\mathbb{P}(L_t = j) = \alpha \Gamma(j - \alpha) / (j! \Gamma(1 - \alpha))$ with $\alpha := e^{-t}$. Thus,

$$g(1, j) = \int_0^\infty \mathbb{P}(L_t = j) dt = \frac{1}{j!} \int_0^1 \frac{\Gamma(j - \alpha)}{\Gamma(1 - \alpha)} d\alpha = \frac{1}{j!} \int_0^1 \frac{\Gamma(j - 1 + x)}{\Gamma(x)} dx$$

and, hence, we obtain the integral representation

$$h(i, j) = h(1, j-i+1) = (j-i+1)g(1, j-i+1) = \frac{1}{(j-i)!} \int_0^1 \frac{\Gamma(j-i+x)}{\Gamma(x)} dx, \quad 1 \leq i \leq j.$$

The last formula for $h(i, j)$ in (10) follows from $\Gamma(n+x)/\Gamma(x) = \sum_{k=0}^n |s(n, k)| x^k$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$. The proof of Corollary 2.3 is complete. \square

Remark. Note that $\mathbb{P}(\eta_1 + \dots + \eta_k = j - i) = \sum_{i_1, \dots, i_k} u_{i_1} \dots u_{i_k}$, where the sum extends over all $i_1, \dots, i_k \in \mathbb{N}$ satisfying $i_1 + \dots + i_k = j - i$. Hence, concrete values of the hitting probabilities are $h(1, 1) = 1$, $h(1, 2) = \mathbb{P}(\eta_1 = 1) = u_1 = 1/2$, $h(1, 3) = \mathbb{P}(\eta_1 = 2) + \mathbb{P}(\eta_1 + \eta_2 = 2) = u_2 + u_1^2 = 1/6 + 1/4 = 5/12 \approx 0.41667$, $h(1, 4) = \mathbb{P}(\eta_1 = 3) + \mathbb{P}(\eta_1 + \eta_2 = 3) + \mathbb{P}(\eta_1 + \eta_2 + \eta_3 = 3) = u_3 + 2u_1u_2 + u_1^3 = 1/12 + 1/6 + 1/8 = 3/8 = 0.375$, $h(1, 5) = u_4 + (2u_1u_3 + u_2^2) + 3u_1^2u_2 = 1/20 + 1/9 + 1/8 = 251/720 \approx 0.34861$, $h(1, 6) = 95/288 \approx 0.32986$, $h(1, 7) = 19087/60480 \approx 0.31559$ and so on.

Proof. (of Corollary 2.4) By the definition of τ_{ni} and the duality of $(N_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ we have $\mathbb{P}(\tau_{ni} \leq t) = \mathbb{P}(N_t^{(n)} \leq i) = \mathbb{P}(L_t^{(i)} \geq n) = \sum_{j=n}^{\infty} p_{ij}(t)$. Using the second formula for $p_{ij}(t)$ in (3) yields

$$\begin{aligned} \mathbb{P}(\tau_{ni} \leq t) &= \sum_{j=n}^{\infty} (-1)^j \sum_{k=1}^i (-1)^k \binom{i}{k} \binom{e^{-t}k}{j} \\ &= \sum_{k=1}^i (-1)^k \binom{i}{k} \sum_{j=n}^{\infty} (-1)^j \binom{e^{-t}k}{j} \\ &= \sum_{k=1}^i (-1)^k \binom{i}{k} (-1)^n \binom{e^{-t}k - 1}{n - 1}, \end{aligned}$$

where the last equality holds since $\sum_{j=n}^{\infty} (-1)^j \binom{z}{j} = (-1)^n \binom{z-1}{n-1}$ for all $n \in \mathbb{N}$ and all $z \in \mathbb{R}$. Fix $x \in \mathbb{R}$ and define $F(x) := e^{-e^{-x}}$ for convenience. Assume that n is sufficiently large such that $x + \log \log n > 0$. Choosing $t := x + \log \log n$ and noting that for all sufficiently large n

$$\begin{aligned} (-1)^{n-1} \binom{e^{-t}k - 1}{n - 1} &= \frac{\Gamma(n - ke^{-x}/\log n)}{\Gamma(n)\Gamma(1 - ke^{-x}/\log n)} \\ &\sim \frac{\Gamma(n - ke^{-x}/\log n)}{\Gamma(n)} \rightarrow e^{-ke^{-x}} = (F(x))^k \end{aligned}$$

as $n \rightarrow \infty$ by an application of Stirling's formula $\Gamma(n+1) \sim (n/e)^n \sqrt{2\pi n}$ as $n \rightarrow \infty$, it follows that

$$\begin{aligned} \mathbb{P}(\tau_{ni} - \log \log n \leq x) &= \mathbb{P}(\tau_{ni} \leq x + \log \log n) \\ &\rightarrow \sum_{k=1}^i (-1)^{k-1} \binom{i}{k} (F(x))^k = 1 - (1 - F(x))^i, \quad n \rightarrow \infty. \end{aligned}$$

It remains to note that $x \mapsto 1 - (1 - F(x))^i$, $x \in \mathbb{R}$, is the distribution function of the minimum of i standard Gumbel distributed random variables. \square

Before we will prove Corollary 2.5 we provide the Taylor expansion of the map $x \mapsto 1/\Gamma(1-x)$.

Lemma 4.1 *The map $x \mapsto 1/\Gamma(1-x)$ has Taylor expansion $1/\Gamma(1-x) = \sum_{k=0}^{\infty} c_k x^k$, $|x| < 1$, where the coefficients c_0, c_1, \dots are related to the moments $m_k = (-1)^k \Gamma^{(k)}(1)$, $k \in \mathbb{N}_0$, of the Gumbel distribution via $c_0 = m_0 = 1$ and*

$$c_k = \sum_{j=1}^k (-1)^j \sum_{\substack{k_1, \dots, k_j \in \mathbb{N} \\ k_1 + \dots + k_j = k}} \frac{m_{k_1} \dots m_{k_j}}{k_1! \dots k_j!}, \quad k \in \mathbb{N}. \quad (22)$$

Alternatively,

$$c_k = \frac{(-1)^k}{k!} \sum_{l=1}^k (-1)^l \binom{k+1}{l+1} (\Gamma^l)^{(k)}(1) \quad k \in \mathbb{N}, \quad (23)$$

where $(\Gamma^l)^{(k)}$ denotes the k th derivative of the l th power of Γ .

Remark. Concrete values are $c_1 = -m_1 = -\gamma \approx -0.577216$, $c_2 = m_1^2 - m_2/2 = \gamma^2 - (\gamma^2 + \zeta(2))/2 = \gamma^2/2 - \pi^2/12 \approx -0.655878$, $c_3 = -m_3/6 + m_1m_2 - m_1^3 = \gamma\zeta(2)/2 - \zeta(3)/3 - \gamma^3/6 = \pi^2\gamma/12 - \zeta(3)/3 - \gamma^3/6 \approx 0.042003$ and so on.

Proof. A Gumbel distributed random variable τ has moment generating function $\mathbb{E}(e^{x\tau}) = \Gamma(1-x)$, $x < 1$. Thus, the map $x \mapsto \Gamma(1-x)$ has Taylor expansion $\Gamma(1-x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < 1$, where $a_k := m_k/k!$ and $m_k = \mathbb{E}(\tau^k)$, $k \in \mathbb{N}_0$, are the moments of the Gumbel distribution. For the reciprocal map $1/\Gamma(1-x)$ it follows that

$$\begin{aligned} \frac{1}{\Gamma(1-x)} &= \sum_{j=0}^{\infty} (1-\Gamma(1-x))^j = \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} -a_k x^k \right)^j \\ &= 1 + \sum_{j=1}^{\infty} \sum_{k_1, \dots, k_j \in \mathbb{N}} (-a_{k_1}) \cdots (-a_{k_j}) x^{k_1 + \dots + k_j} \\ &= 1 + \sum_{j=1}^{\infty} (-1)^j \sum_{k=1}^{\infty} x^k \sum_{\substack{k_1, \dots, k_j \in \mathbb{N} \\ k_1 + \dots + k_j = k}} a_{k_1} \cdots a_{k_j} = \sum_{k=0}^{\infty} c_k x^k \end{aligned}$$

with $c_0 := 1$ and c_k , $k \in \mathbb{N}$, as given in (22), since $a_k = m_k/k!$, $k \in \mathbb{N}_0$. Since $m_k = (-1)^k \Gamma^{(k)}(1)$, (22) can be rewritten as

$$\begin{aligned} c_k &= \sum_{j=1}^k (-1)^{j+k} \sum_{\substack{k_1, \dots, k_j \in \mathbb{N} \\ k_1 + \dots + k_j = k}} \frac{\Gamma^{(k_1)}(1) \cdots \Gamma^{(k_j)}(1)}{k_1! \cdots k_j!} \\ &= \sum_{j=1}^k \frac{(-1)^{j+k}}{k!} \sum_{l=1}^j (-1)^{j-l} \binom{j}{l} (\Gamma^{(l)})^{(k)}(1), \quad k \in \mathbb{N}, \end{aligned}$$

where the last equality holds by Lemma 1 in the appendix of [20]. Interchanging the sums and noting that $\sum_{j=l}^k \binom{j}{l} = \binom{k+1}{l+1}$ yields (23). \square

Proof. (of Corollary 2.5) Fix $x \in \mathbb{R}$ and define $F(x) := e^{-e^{-x}}$. By Corollary 2.4, for all sufficiently large n ,

$$\mathbb{P}(\tau_{ni} - \log \log n \leq x) = \sum_{j=1}^i (-1)^{j-1} \binom{i}{j} \frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n) \Gamma(1 - je^{-x}/\log n)}. \quad (24)$$

For every $c \in \mathbb{R}$ it is easily checked that $\Gamma(n + c/\log n)/\Gamma(n) = e^c + O(1/(n \log n))$ as $n \rightarrow \infty$. For $c = -je^{-x}$ we obtain

$$\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)} = (F(x))^j + O\left(\frac{1}{n \log n}\right). \quad (25)$$

Moreover (see Lemma 4.1), from $1/\Gamma(1-x) = \sum_{k=0}^{\infty} c_k x^k$ we conclude that, for all $K \in \mathbb{N}_0$,

$$\frac{1}{\Gamma(1 - je^{-x}/\log n)} = \sum_{k=0}^K c_k \left(\frac{je^{-x}}{\log n} \right)^k + O\left(\frac{1}{(\log n)^{K+1}}\right). \quad (26)$$

Multiplying (25) with (26) yields

$$\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n) \Gamma(1 - je^{-x}/\log n)} = (F(x))^j \sum_{k=0}^K c_k \left(\frac{je^{-x}}{\log n} \right)^k + O\left(\frac{1}{(\log n)^{K+1}}\right).$$

Plugging this expansion into (24) and exchanging the sums yields

$$\mathbb{P}(\tau_{ni} - \log \log n \leq x) = \sum_{k=0}^K c_k \left(\frac{e^{-x}}{\log n} \right)^k \sum_{j=1}^i (F(x))^j (-1)^{j-1} \binom{i}{j} j^k + O\left(\frac{1}{(\log n)^{K+1}}\right),$$

which is the desired Edgeworth expansion with coefficients $d_{ki}(x)$ as defined in (13). It remains to verify the alternative representation (14) of the coefficients $d_{ki}(x)$. It is readily checked by induction on $k \in \mathbb{N}_0$ that $(t \frac{\partial}{\partial t})^k f(t) = \sum_{j=0}^k S(k, j) t^j f^{(j)}(t)$ for every k -times differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, where the $S(k, j)$ denote the Stirling numbers of the second kind. Applying this formula to $f(t) := 1 - (1 - t)^i$ with $i \in \mathbb{N}$ it follows for all $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$ that

$$\begin{aligned} \sum_{j=1}^i (-1)^{j-1} \binom{i}{j} j^k t^j &= \left(t \frac{\partial}{\partial t} \right)^k \sum_{j=1}^i (-1)^{j-1} \binom{i}{j} t^j = \left(t \frac{\partial}{\partial t} \right)^k (1 - (1 - t)^i) \\ &= \sum_{j=0}^k S(k, j) t^j \left(\frac{\partial}{\partial t} \right)^j (1 - (1 - t)^i) \\ &= S(k, 0)(1 - (1 - t)^i) + \sum_{j=1}^k S(k, j) t^j (-1)^{j-1} (i)_j (1 - t)^{i-j}, \end{aligned}$$

where $(i)_j := i(i-1)\cdots(i-j+1)$. Replacing t by $F(x)$ and noting that $S(k, 0) = 0$ for $k \in \mathbb{N}$ shows that (13) coincides for $k \in \mathbb{N}$ with (14). \square

Proof. (of Theorem 3.1 a)) Let $Z^{(n)} := (X_t^{(n)}, t)_{t \geq 0}$ and $Z := (X_t, t)_{t \geq 0}$ denote the space-time processes of $X^{(n)} = (X_t^{(n)})_{t \geq 0}$ and $X = (X_t)_{t \geq 0}$ respectively. Note that $Z^{(n)}$ has state space $S_n := \{(j/n^{e^{-t}}, t) : j \in \{1, \dots, n\}, t \geq 0\} = \bigcup_{t \geq 0} (E_{n,t} \times \{t\})$, where $E_{n,t} := \{j/n^{e^{-t}} : j \in \{1, \dots, n\}\}$, and that Z has state space $S := E \times [0, \infty) = [0, \infty)^2$. The processes $Z^{(n)}$ and Z are time-homogeneous (see, for example, Revuz and Yor [28, p. 85, Exercise (1.10)]). In the following it is shown that $Z^{(n)}$ converges in $D_S[0, \infty)$ to Z as $n \rightarrow \infty$. Note that this convergence implies the desired convergence of $X^{(n)}$ in $D_E[0, \infty)$ to X as $n \rightarrow \infty$. Define $\pi_n : B(S) \rightarrow B(S_n)$ via $\pi_n f(x, s) := f(x, s)$ for all $f \in B(S)$ and $(x, s) \in S_n$. By Proposition 5.4 it suffices to verify that, for every $t \geq 0$ and $\lambda, \mu > 0$,

$$\lim_{n \rightarrow \infty} \sup_{s \geq 0} \sup_{x \in E_{n,s}} |T_t^{(n)} \pi_n f_{\lambda, \mu}(x, s) - \pi_n T_t f_{\lambda, \mu}(x, s)| = 0,$$

where $(T_t^{(n)})_{t \geq 0}$ and $(T_t)_{t \geq 0}$ denote the semigroups of the space-time processes $Z^{(n)}$ and Z respectively and the test functions $f_{\lambda, \mu} : S \rightarrow \mathbb{R}$ are defined via $f_{\lambda, \mu}(x, s) := e^{-\lambda x - \mu s}$ for all $(x, s) \in S$. Fix $t \geq 0$ and $\lambda, \mu > 0$. For convenience, define $\alpha := e^{-t}$ and $\beta := e^{-s}$. We have

$$\begin{aligned} T_t^{(n)} \pi_n f_{\lambda, \mu}(x, s) &= \mathbb{E}(f_{\lambda, \mu}(X_{s+t}^{(n)}, s+t) \mid X_s^{(n)} = x) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_{s+t}^{(n)}) \mid N_s^{(n)} = xn^\beta) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)})), \quad (x, s) \in S_n, \end{aligned}$$

and

$$\begin{aligned} \pi_n T_t f_{\lambda, \mu}(x, s) &= \mathbb{E}(f_{\lambda, \mu}(X_{s+t}, s+t) \mid X_s = x) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda X_{s+t}) \mid X_s = x) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda x^\alpha X_t)), \quad (x, s) \in S. \end{aligned}$$

Thus, we have to verify that

$$\lim_{n \rightarrow \infty} \sup_{s \geq 0} \sup_{x \in E_{n,s}} |(\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)})) - \mathbb{E}(\exp(-\lambda x^\alpha X_t))| = 0.$$

Since both expectations are bounded between 0 and 1 and since $(\alpha\beta)^\mu = e^{-\mu(s+t)}$ tends to 0 as $s \rightarrow \infty$ it suffices to verify that, for every $s_0 > 0$,

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, s_0]} \sup_{x \in E_{n,s}} |\mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)})) - \mathbb{E}(\exp(-\lambda x^\alpha X_t))| = 0.$$

We will even verify that

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, s_0]} \sup_{x \geq 0} |\mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)})) - \mathbb{E}(\exp(-\lambda x^\alpha X_t))| = 0.$$

The difference of the two expectations depends on n and s only via $n^\beta = n^{e^{-s}}$. Since the map $s \mapsto n^{e^{-s}}$ is non-increasing it follows that the convergence for fixed $s \in [0, s_0]$ is slower as s is larger. So the slowest convergence holds at the right end point $s = s_0$. Thus, it suffices to verify that, for every $s \geq 0$,

$$\lim_{n \rightarrow \infty} \sup_{x \geq 0} |\mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)})) - \mathbb{E}(\exp(-\lambda x^\alpha X_t))| = 0.$$

The map $x \mapsto \mathbb{E}(\exp(-\lambda x^\alpha X_t))$ is bounded, continuous, and non-increasing. Moreover, for every n the map $x \mapsto \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)}))$ is non-increasing. Thus, by the theorem of Pólya, it suffices to verify that, for every $s \geq 0$ and $x \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)})) = \mathbb{E}(\exp(-\lambda x^\alpha X_t)).$$

Note that we have reduced the problem to verify the convergence uniformly for all $s \geq 0$ and $x \in E_{n,s}$ to the problem to verify the convergence pointwise for all points $(s, x) \in [0, \infty)^2$. Define $\tau := n^\beta$. Using this notation it remains to verify that

$$\lim_{\tau \rightarrow \infty} \mathbb{E}(\exp(-\lambda/\tau^\alpha N_t^{(\lfloor x\tau \rfloor)})) = \mathbb{E}(\exp(-\lambda x^\alpha X_t)). \quad (27)$$

We have

$$\mathbb{E}(\exp(-\lambda/\tau^\alpha N_t^{(\lfloor x\tau \rfloor)})) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} \frac{\mathbb{E}((N_t^{(\lfloor x\tau \rfloor)})^m)}{\tau^{\alpha m}}.$$

Note that the series on the right hand side is absolutely convergent, since $N_t^{(\lfloor x\tau \rfloor)} \leq x\tau$ and, hence, $\mathbb{E}((N_t^{(\lfloor x\tau \rfloor)})^m) \leq (x\tau)^m$. Applying the formula $z^m = \sum_{i=0}^m (-1)^{m-i} S(m, i) [z]_i$, $m \in \mathbb{N}_0$, $z > 0$, where $[z]_i := \Gamma(z+i)/\Gamma(z)$ for $z, i > 0$, it follows that

$$\frac{\mathbb{E}((N_t^{(\lfloor x\tau \rfloor)})^m)}{\tau^{\alpha m}} = \sum_{i=0}^m (-1)^{m-i} S(m, i) \frac{\mathbb{E}([N_t^{(\lfloor x\tau \rfloor)]}_i)}{\tau^{\alpha m}} = \sum_{i=0}^m (-1)^{m-i} S(m, i) \mathbb{E}(X_t^i) \frac{[x\tau]_{\alpha i}}{\tau^{\alpha m}}$$

by Lemma 3.1 of [21]. From $[x\tau]_{\alpha i} \sim (x\tau)^{\alpha i} = x^{\alpha i} \tau^{\alpha i}$ as $\tau \rightarrow \infty$ we conclude that only the summand $i = m$ yields asymptotically a non-zero contribution and it follows that

$$\lim_{\tau \rightarrow \infty} \frac{\mathbb{E}((N_t^{(\lfloor x\tau \rfloor)})^m)}{\tau^{\alpha m}} = \mathbb{E}(X_t^m) x^{\alpha m} = \mathbb{E}((x^\alpha X_t)^m).$$

Moreover,

$$\frac{\mathbb{E}((N_t^{(\lfloor x\tau \rfloor)})^m)}{\tau^{\alpha m}} \leq \frac{\mathbb{E}([N_t^{(\lfloor x\tau \rfloor)]}_m)}{\tau^{\alpha m}} = \mathbb{E}(X_t^m) \frac{[x\tau]_{\alpha m}}{\tau^{\alpha m}} \leq \mathbb{E}(X_t^m) \frac{[x\tau]_{\alpha m}}{\tau^{\alpha m}}.$$

It is readily checked that the map $\tau \mapsto [x\tau]_{\alpha m}/\tau^{\alpha m}$ is non-increasing in τ . Thus, we obtain the upper bound

$$\frac{\mathbb{E}((N_t^{(\lfloor x\tau \rfloor)})^m)}{\tau^{\alpha m}} \leq \mathbb{E}(X_t^m) \frac{[x\tau_0]_{\alpha m}}{\tau_0^{\alpha m}} \quad \text{for all } \tau \geq \tau_0.$$

Note that

$$\frac{\lambda^m}{m!} \mathbb{E}(X_t^m) \frac{[x\tau_0]_{\alpha m}}{\tau_0^{\alpha m}} = \frac{\lambda^m}{m!} \frac{m!}{\Gamma(1+\alpha m)} \frac{\Gamma(x\tau_0 + \alpha m)}{\tau_0^{\alpha m} \Gamma(x\tau_0)} \sim \left(\frac{\lambda}{\tau_0^\alpha}\right)^m (\alpha m)^{x\tau_0-1}$$

as $m \rightarrow \infty$. Thus, if we choose τ_0 sufficiently large such that $\lambda/\tau_0^\alpha < 1$, for example $\tau_0 := (2\lambda)^{1/\alpha}$, then the dominating map $m \mapsto (\lambda^m/m!) \mathbb{E}(X_t^m)[x\tau_0]_{\alpha m}/\tau_0^{\alpha m}$ is integrable with respect to the counting measure on \mathbb{N} . Thus, it is allowed to apply the dominated convergence theorem, which yields

$$\lim_{\tau \rightarrow \infty} \mathbb{E}(\exp(-\lambda/\tau^\alpha N_t^{(\lfloor x\tau \rfloor)})) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} \mathbb{E}((x^\alpha X_t)^m) = \mathbb{E}(\exp(-\lambda x^\alpha X_t)).$$

Thus, (27) is established. The proof is complete. \square

Before we come to the proof of Theorem 3.1 b), we provide a recursion for the Laplace transforms of the finite-dimensional distributions of Neveu's continuous-state branching process $Y = (Y_t)_{t \geq 0}$.

Lemma 4.2 (Recursion for the Laplace transforms of Y) *Let $0 = t_0 \leq t_1 < t_2 < \dots$. For $k \in \mathbb{N}$ let $\psi_k : [0, \infty)^k \rightarrow [0, 1]$, defined via $\psi_k(\lambda_1, \dots, \lambda_k) := \mathbb{E}(e^{-\lambda_1 Y_{t_1}} \dots e^{-\lambda_k Y_{t_k}})$ for all $\lambda_1, \dots, \lambda_k \geq 0$, denote the Laplace transform of Y_{t_1}, \dots, Y_{t_k} . Then, ψ_k satisfies the recursion $\psi_1(\lambda_1) = e^{-\lambda_1^{\alpha_1}}$ for all $\lambda_1 \geq 0$ and*

$$\psi_k(\lambda_1, \dots, \lambda_k) = \psi_{k-1}(\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}), \quad k \in \mathbb{N} \setminus \{1\}, \lambda_1, \dots, \lambda_k \geq 0,$$

where $\alpha_j := e^{-t_j}$, $1 \leq j \leq k$.

Proof. (of Lemma 4.2) Clearly, $\psi_1(\lambda_1) = \mathbb{E}(e^{-\lambda_1 Y_{t_1}}) = e^{-\lambda_1^{\alpha_1}}$ for all $\lambda_1 \geq 0$. Moreover, for all $\lambda_1, \dots, \lambda_k \geq 0$,

$$\begin{aligned} \psi_k(\lambda_1, \dots, \lambda_k) &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}} \dots e^{-\lambda_k Y_{t_k}}) \\ &= \mathbb{E}(\mathbb{E}(e^{-\lambda_1 Y_{t_1}} \dots e^{-\lambda_k Y_{t_k}} | Y_{t_1}, \dots, Y_{t_{k-1}})) \\ &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}} \dots e^{-\lambda_{k-1} Y_{t_{k-1}}} \mathbb{E}(e^{-\lambda_k Y_{t_k}} | Y_{t_{k-1}})). \end{aligned}$$

Since $\mathbb{E}(e^{-\lambda_k Y_{t_k}} | Y_{t_{k-1}}) = e^{-\lambda_k^{\alpha_k/\alpha_{k-1}} Y_{t_{k-1}}}$ almost surely it follows that

$$\begin{aligned} \psi_k(\lambda_1, \dots, \lambda_k) &= \mathbb{E}(e^{\lambda_1 Y_{t_1}} \dots e^{-\lambda_{k-2} Y_{t_{k-2}}} e^{-(\lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}) Y_{t_{k-1}}}) \\ &= \psi_{k-1}(\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}). \end{aligned} \quad \square$$

We are now able to verify Theorem 3.1 b).

Proof. (of Theorem 3.1 b)) The proof is divided into two parts. First the convergence of the finite-dimensional distributions is verified. Afterwards the convergence in $D_E[0, \infty)$ is considered. In fact Part 2 does not use results from Part 1, so one could omit Part 1. However, we think it is helpful for the reader to consider first the convergence of the finite-dimensional distributions.

Part 1. (Convergence of the finite-dimensional distributions) Fix $0 = t_0 \leq t_1 < t_2 < \dots$. For $k, n \in \mathbb{N}$ let $\psi_k^{(n)} : [0, \infty)^k \rightarrow [0, 1]$ and $\psi_k : [0, \infty)^k \rightarrow [0, 1]$ denote the Laplace transforms of $(Y_{t_1}^{(n)}, \dots, Y_{t_k}^{(n)})$ and $(Y_{t_1}, \dots, Y_{t_k})$ respectively. In the following the pointwise convergence $\psi_k^{(n)} \rightarrow \psi_k$ as $n \rightarrow \infty$ is verified by induction on $k \in \mathbb{N}$.

Obviously, $L_{t_1}^{(n)}$ has generating function $\mathbb{E}(z_1^{L_{t_1}^{(n)}}) = (1 - (1 - z_1)^{\alpha_1})^n$, $z_1 \in [0, 1]$, where $\alpha_1 := e^{-t_1}$. Replacing z_1 by $e^{-\lambda_1/n^{1/\alpha_1}}$ with $\lambda_1 \geq 0$ it follows that

$$\psi_1^{(n)}(\lambda_1) = \mathbb{E}(e^{-\lambda_1 Y_{t_1}^{(n)}}) = (1 - (1 - e^{-\lambda_1/n^{1/\alpha_1}})^{\alpha_1})^n.$$

Clearly, $\psi_1(\lambda_1) = \mathbb{E}(e^{-\lambda_1 Y_{t_1}}) = e^{-\lambda_1^{\alpha_1}}$. Using the shortage $x := \lambda_1/n^{1/\alpha_1}$ and the inequality $|a^n - b^n| \leq n|a - b|$, $|a|, |b| \leq 1$, it follows that

$$\begin{aligned} |\psi_1^{(n)}(\lambda_1) - \psi_1(\lambda_1)| &= |(1 - (1 - e^{-x})^{\alpha_1})^n - (e^{-x^{\alpha_1}})^n| \\ &\leq n|1 - (1 - e^{-x})^{\alpha_1} - e^{-x^{\alpha_1}}| = n(e^{-x^{\alpha_1}} - 1 + (1 - e^{-x})^{\alpha_1}), \end{aligned}$$

since $(1 - e^{-x})^{\alpha_1} \geq 1 - e^{-x^{\alpha_1}}$ by Lemma 5.1. From $1 - e^{-x} \leq x$, $x \in \mathbb{R}$, and $e^{-t} - 1 + t \leq t^2/2$, $t \geq 0$, we conclude that

$$|\psi_1^{(n)}(\lambda_1) - \psi_1(\lambda_1)| \leq n(e^{-x^{\alpha_1}} - 1 + x^{\alpha_1}) \leq n \frac{(x^{\alpha_1})^2}{2} = \frac{\lambda_1^{2\alpha_1}}{2n} \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, the pointwise convergence $\psi_1^{(n)} \rightarrow \psi_1$ as $n \rightarrow \infty$ is established.

Now fix $k \in \mathbb{N} \setminus \{1\}$. The induction step from $k-1$ to k works as follows. For convenience define $\alpha_j := e^{-t_j}$ for all $j \in \mathbb{N}$. For all $z_1, \dots, z_k \in [0, 1]$,

$$\begin{aligned} \mathbb{E}(z_1^{L_{t_1}^{(n)}} \dots z_k^{L_{t_k}^{(n)}}) &= \mathbb{E}(\mathbb{E}(z_1^{L_{t_1}^{(n)}} \dots z_k^{L_{t_k}^{(n)}} | L_{t_1}^{(n)}, \dots, L_{t_{k-1}}^{(n)})) \\ &= \mathbb{E}(z_1^{L_{t_1}^{(n)}} \dots z_{k-1}^{L_{t_{k-1}}^{(n)}} \mathbb{E}(z_k^{L_{t_k}^{(n)}} | L_{t_{k-1}}^{(n)})). \end{aligned}$$

Since $\mathbb{E}(z_k^{L_{t_k}^{(n)}} | L_{t_{k-1}}^{(n)}) = (1 - (1 - z_k)^{\alpha_k/\alpha_{k-1}})^{L_{t_{k-1}}^{(n)}}$ almost surely it follows that

$$\mathbb{E}(z_1^{L_{t_1}^{(n)}} \dots z_k^{L_{t_k}^{(n)}}) = \mathbb{E}(z_1^{L_{t_1}^{(n)}} \dots z_{k-2}^{L_{t_{k-2}}^{(n)}} u_{k-1}^{L_{t_{k-1}}^{(n)}}),$$

where $u_{k-1} := z_{k-1}(1 - (1 - z_k)^{\alpha_k/\alpha_{k-1}})$. Replacing for each $j \in \{1, \dots, k\}$ the variable z_j by $e^{-\lambda_j/n^{1/\alpha_j}}$ with $\lambda_j \geq 0$ it follows that

$$\begin{aligned} \psi_k^{(n)}(\lambda_1, \dots, \lambda_k) &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}^{(n)}} \dots e^{-\lambda_k Y_{t_k}^{(n)}}) \\ &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}^{(n)}} \dots e^{-\lambda_{k-2} Y_{t_{k-2}}^{(n)}} e^{-\mu_{k-1}(n) Y_{t_{k-1}}^{(n)}}) \\ &= \psi_{k-1}^{(n)}(\lambda_1, \dots, \lambda_{k-2}, \mu_{k-1}(n)), \end{aligned} \tag{28}$$

where

$$\mu_{k-1}(n) := \lambda_{k-1} - n^{1/\alpha_{k-1}} \log(1 - (1 - e^{-\lambda_k/n^{1/\alpha_k}})^{\alpha_k/\alpha_{k-1}}).$$

A technical but straightforward calculation shows that $\mu_{k-1}(n) \rightarrow \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}$ as $n \rightarrow \infty$. Moreover, by induction, $\psi_{k-1}^{(n)}$ converges pointwise to ψ_{k-1} as $n \rightarrow \infty$. It is well known that the convergence $\psi_{k-1}^{(n)} \rightarrow \psi_{k-1}$ of Laplace transforms holds even uniformly on any compact subset of $[0, \infty)^{k-1}$. Taking these facts into account it follows from (28) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_k^{(n)}(\lambda_1, \dots, \lambda_k) &= \lim_{n \rightarrow \infty} \psi_{k-1}^{(n)}(\lambda_1, \dots, \lambda_{k-2}, \mu_{k-1}(n)) \\ &= \psi_{k-1}(\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}) = \psi_k(\lambda_1, \dots, \lambda_k), \end{aligned}$$

where the last equality holds by Lemma 4.2. The induction is complete.

The pointwise convergence $\psi_k^{(n)} \rightarrow \psi_k$ of the Laplace transforms implies the convergence $(Y_{t_1}^{(n)}, \dots, Y_{t_k}^{(n)}) \rightarrow (Y_{t_1}, \dots, Y_{t_k})$ in distribution as $n \rightarrow \infty$.

Part 2. (Convergence in $D_E[0, \infty)$) Recall that $E := [0, \infty)$ is the state space of the limiting process Y . For $n \in \mathbb{N}$ and $t \geq 0$ define $E_{n,t} := \{j/n^{e^t} : j = n, n+1, \dots\}$. Note that the processes $Y^{(n)}$ are time-inhomogeneous. In order to obtain time-homogeneous processes let $Z^{(n)} := (Y_t^{(n)}, t)_{t \geq 0}$ and $Z := (Y_t, t)_{t \geq 0}$ denote the space-time processes of $(Y_t^{(n)})_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ respectively. Note that $Z^{(n)}$ has state space $S_n := \{(j/n^{e^t}, t) : j = n, n+1, \dots, t \geq 0\} = \bigcup_{t \geq 0} (E_{n,t} \times \{t\})$ and that Z has state space $S := E \times [0, \infty) = [0, \infty)^2$. According to Revuz and Yor [28, p. 85, Exercise (1.10)], the processes $Z^{(n)}$ and Z are time-homogeneous. Define $\pi_n : B(S) \rightarrow B(S_n)$ via $\pi_n g(y, s) := g(y, s)$ for all $g \in B(S)$ and $(y, s) \in S_n$. In the following it is shown that $Z^{(n)}$ converges in $D_S[0, \infty)$ to Z as $n \rightarrow \infty$. For $\lambda, \mu > 0$ define the test function $g_{\lambda, \mu} \in \widehat{C}(S)$ via $g_{\lambda, \mu}(y, s) := e^{-\lambda y - \mu s}$, $(y, s) \in S$. By Proposition 5.4 it suffices to verify that for every $t \geq 0$ and $\lambda, \mu > 0$,

$$\lim_{n \rightarrow \infty} \sup_{s \geq 0} \sup_{y \in E_{n,s}} |U_t^{(n)} \pi_n g_{\lambda, \mu}(y, s) - \pi_n U_t g_{\lambda, \mu}(y, s)| = 0, \tag{29}$$

where $U_t^{(n)} : B(S_n) \rightarrow B(S_n)$ is defined via $U_t^{(n)} g(y, s) := \mathbb{E}(g(Y_{s+t}^{(n)}, s+t) | Y_s^{(n)} = y)$, $g \in B(S_n)$, $s \geq 0$, $y \in E_{n,s}$. Note that $(U_t^{(n)})_{t \geq 0}$ is the semigroup of $Z^{(n)}$. Fix $t \geq 0$ and $\lambda, \mu > 0$. As before define $\alpha := e^{-t}$ and $\beta := e^{-s}$. For all $n \in \mathbb{N}$, $s \geq 0$ and $y \in E_{n,s}$,

$$\begin{aligned} U_t^{(n)} \pi_n g_{\lambda, \mu}(y, s) &= \mathbb{E}(\pi_n g_{\lambda, \mu}(Y_{s+t}^{(n)}, s+t) | Y_s^{(n)} = y) \\ &= \mathbb{E}(\exp(-\lambda Y_{s+t}^{(n)} - \mu(s+t)) | Y_s^{(n)} = y) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)} L_{s+t}^{(n)}) | L_s^{(n)} = yn^{1/\beta}) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(yn^{1/\beta})})) \\ &= (\alpha\beta)^\mu (1 - (1 - e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha) yn^{1/\beta} \end{aligned}$$

and $\pi_n U_t g_{\lambda, \mu}(y, s) = U_t g_{\lambda, \mu}(y, s) = \mathbb{E}(\exp(-\lambda Y_{s+t} - \mu(s+t)) | Y_s = y) = (\alpha\beta)^\mu e^{-y\lambda^\alpha}$. Define $m := yn^{1/\beta} \in \{n, n+1, \dots\}$ and $x := \lambda/n^{1/(\alpha\beta)}$. In the following it is assumed that $n \geq \lambda$ which implies that $x \leq 1$. Using the inequality $|a^m - b^m| \leq mr^{m-1}|a - b|$, $m \in \mathbb{N}$, where $r := \max(|a|, |b|)$, it follows that

$$\begin{aligned} d &:= |(1 - (1 - e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha) yn^{1/\beta} - e^{-y\lambda^\alpha}| \\ &= |(1 - (1 - e^{-x})^\alpha)^m - (e^{-x^\alpha})^m| \\ &\leq mr^{m-1} |1 - (1 - e^{-x})^\alpha - e^{-x^\alpha}|, \end{aligned}$$

where $r := \max(1 - (1 - e^{-x})^\alpha, e^{-x^\alpha}) = e^{-x^\alpha}$ by Lemma 5.1. Note that $r \in (0, 1)$. The map $z \mapsto zr^{z-1}$, $z \geq 0$ takes its maximum at the point $z = 1/(-\log r) = 1/x^\alpha$. Thus, $mr^{m-1} \leq 1/x^\alpha r^{1/x^\alpha - 1} \leq 1/x^\alpha$, since $r \leq 1$ and $x \leq 1$, i.e. $1/x^\alpha - 1 \geq 0$. Furthermore, $|1 - (1 - e^{-x})^\alpha - e^{-x^\alpha}| = e^{-x^\alpha} - 1 + (1 - e^{-x})^\alpha \leq e^{-x^\alpha} - 1 + x^\alpha \leq (x^\alpha)^2/2$. Therefore, we obtain the upper bound

$$d \leq \frac{1}{x^\alpha} \frac{(x^\alpha)^2}{2} = \frac{x^\alpha}{2} = \frac{\lambda^\alpha}{2ne^s} \leq \frac{\lambda^\alpha}{2n}.$$

Note that this upper bound does not depend on y and s . Thus, for all $t \geq 0$, $\lambda, \mu > 0$ and all $n \in \mathbb{N}$ with $n \geq \lambda$,

$$\begin{aligned} &\sup_{s \geq 0} \sup_{y \in E_{n,s}} |U_t^{(n)} \pi_n g_{\lambda, \mu}(y, s) - \pi_n U_t g_{\lambda, \mu}(y, s)| \\ &= \sup_{s \geq 0} \sup_{y \in E_{n,s}} \underbrace{|e^{-\mu(s+t)}|}_{\leq 1} |(1 - (1 - e^{-\lambda/n^{e^{s+t}}})e^{-t})^{yn^{e^s}} - e^{-y\lambda^{e^{-t}}}| \leq \frac{\lambda^\alpha}{2n} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore, (29) holds for all $t \geq 0$ and all $\lambda, \mu > 0$. \square

5 Appendix

Lemma 5.1 *For all $x \geq 0$ and all $\alpha \in [0, 1]$ we have $(1 - e^{-x})^\alpha \geq 1 - e^{-x^\alpha}$.*

Proof. Fix $\alpha \in [0, 1]$. If $x \geq 1$ then $x^\alpha \leq x$ and, hence, $(1 - e^{-x})^\alpha \geq 1 - e^{-x} \geq 1 - e^{-x^\alpha}$. Assume now that $x \in [0, 1]$. Then $x^\alpha \geq x$. The function $f(x) := (1 - e^{-x})^\alpha - 1 + e^{-x^\alpha}$ satisfies $f(0) = 0$ and has derivative $f'(x) = \alpha e^{-x}(1 - e^{-x})^{\alpha-1} - \alpha x^{\alpha-1} e^{-x^\alpha}$, which is nonnegative on $[0, 1]$, since $e^{-x} \geq e^{-x^\alpha}$ and $(1 - e^{-x})^{\alpha-1} \geq x^{\alpha-1}$ for $x \in [0, 1]$. From $f(0) = 0$ and $f'(x) \geq 0$ for $x \in [0, 1]$ it follows that $f(x) \geq 0$ for $x \in [0, 1]$, which is the desired inequality. \square

Lemma 5.2 (Spectral decomposition of Γ for the Kingman coalescent)

The generator $\Gamma = (\gamma_{ij})_{i,j \in \mathbb{N}}$ of the fixation line $(L_t)_{t \geq 0}$ of the Kingman coalescent has spectral decomposition $\Gamma = RDL$, where $D = (d_{ij})_{i,j \in \mathbb{N}}$ is the diagonal matrix with entries

$d_{ij} = -i(i+1)/2$ for $i = j$ and $d_{ij} = 0$ for $i \neq j$, and $R = (r_{ij})_{i,j \in \mathbb{N}}$ and $L = (l_{ij})_{i,j \in \mathbb{N}}$ are upper right triangular matrices with entries

$$r_{ij} = (-1)^{j-i} \frac{j!(j-1)!(i+j)!}{(j-i)!i!(i-1)!(2j)!}, \quad i, j \in \mathbb{N}, i \leq j, \quad (30)$$

and

$$l_{ij} = \frac{j!(j-1)!(2i+1)!}{i!(i-1)!(j-i)!(i+j+1)!}, \quad i, j \in \mathbb{N}, i \leq j. \quad (31)$$

Remark. Note that $l_i(z) := \sum_{j=i}^{\infty} l_{ij} z^{j+1}$ satisfies the differential equation $z^2(1-z)l_i''(z) = i(i+1)l_i(z)$, $i \in \mathbb{N}$, $|z| < 1$.

Proof. For a pure birth process the recursion (20) reduces to $r_{ij} = \gamma_i/(\gamma_i - \gamma_j)r_{i+1,j}$, $i \in \{j-1, j-2, \dots, 1\}$, with solution $r_{ij} = \prod_{k=i}^{j-1} \gamma_k/(\gamma_k - \gamma_j)$, $i \leq j$. Thus, for the Kingman coalescent, for all $i, j \in \mathbb{N}$ with $i \leq j$,

$$r_{ij} = \prod_{k=i}^{j-1} \frac{k(k+1)}{k(k+1) - j(j+1)} = \prod_{k=i}^{j-1} \frac{k(k+1)}{(k-j)(k+j+1)} = (-1)^{j-i} \frac{j!(j-1)!(i+j)!}{(j-i)!i!(i-1)!(2j)!}.$$

Similarly, the recursion (21) reduces to $l_{ij} = \gamma_{j-1}/(\gamma_j - \gamma_i)l_{i,j-1}$, $j \in \{i+1, i+2, \dots\}$, with solution $l_{ij} = \prod_{k=i+1}^j \gamma_{k-1}/(\gamma_k - \gamma_i)$, $i \leq j$. Thus, for the Kingman coalescent, for all $i, j \in \mathbb{N}$ with $i \leq j$,

$$l_{ij} = \prod_{k=i+1}^j \frac{k(k-1)}{k(k+1) - i(i+1)} = \prod_{k=i+1}^j \frac{k(k-1)}{(k-i)(k+i+1)} = \frac{j!(j-1)!(2i+1)!}{i!(i-1)!(j-i)!(i+j+1)!}.$$

□

Let E be locally compact, i.e. every point $x \in E$ has a compact neighborhood. A function $f : E \rightarrow \mathbb{R}$ vanishes at infinity, if for every $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $|f(x)| < \varepsilon$ for all $x \in E \setminus K$. In other words $\{x \in E : |f(x)| \geq \varepsilon\}$ is compact. In the following $\widehat{C}(E)$ denotes the set of all real-valued continuous functions on E vanishing at infinity.

Lemma 5.3 *Let $d \in \mathbb{N}$. The set D of all functions $g : [0, \infty)^d \rightarrow \mathbb{R}$ of the form $g(y) = \sum_{i_1, \dots, i_d=1}^m a_{i_1, \dots, i_d} e^{-(i_1 y_1 + \dots + i_d y_d)}$ with $m \in \mathbb{N}$ and $a_{i_1, \dots, i_d} \in \mathbb{R}$ is dense in $\widehat{C}([0, \infty)^d)$.*

Proof. Let $g \in \widehat{C}([0, \infty)^d)$. Define $f : [0, 1]^d \rightarrow \mathbb{R}$ via $f(x) := g(-\log x_1, \dots, -\log x_d)$ for $x \in (0, 1]^d$ and $f(x) := 0$ if $x_j = 0$ for some $j \in \{1, \dots, d\}$. Since g is continuous and vanishes at infinity it follows that f is continuous. For $n \in \mathbb{N}$ and $x = (x_1, \dots, x_d) \in [0, 1]^d$ let

$$p_n(x) := \sum_{k_1, \dots, k_d=1}^n f\left(\frac{k_1}{n}, \dots, \frac{k_d}{n}\right) \prod_{j=1}^d \binom{n}{k_j} x_j^{k_j} (1-x_j)^{n-k_j}.$$

denote the n th multivariate Bernstein polynomial of f . Note that the sum runs only over $k = (k_1, \dots, k_d) \in \{1, \dots, n\}^d$ (not as usual over $k \in \{0, \dots, n\}^d$) since $f(x) = 0$ if $x_j = 0$ for some $j \in \{1, \dots, d\}$. By a d -dimensional version of Bernstein's approximation theorem (see, for example, [7, Theorem 8]), $p_n \rightarrow f$ as $n \rightarrow \infty$ uniformly on $[0, 1]^d$. Replacing x_j by e^{-y_j} it follows that $g_n \rightarrow g$ as $n \rightarrow \infty$ uniformly on $[0, \infty)^d$, where $g_n(y) := p_n(e^{-y_1}, \dots, e^{-y_d})$. It remains to note that $g_n \in D$. □

Proposition 5.4 (Convergence of Markov processes) *Let $d \in \mathbb{N}$, $E := [0, \infty)^d$ and $X = (X_t)_{t \geq 0}$ be an E -valued time-homogeneous Markov process. Furthermore, for every $n \in \mathbb{N}$ let $X^{(n)} = (X_t^{(n)})_{t \geq 0}$ be an E_n -valued time-homogeneous Markov process with state space $E_n \subseteq E$. Let $(T_t)_{t \geq 0}$ and $(T_t^{(n)})_{t \geq 0}$ denote the corresponding semigroups. Define $\pi_n :$*

$B(E) \rightarrow B(E_n)$ via $\pi_n f(x) := f(x)$ for all $f \in B(E)$ and $x \in E_n$. If, for every $t \geq 0$ and $\lambda \in \mathbb{N}^d$,

$$\lim_{n \rightarrow \infty} \|T_t^{(n)} \pi_n f_\lambda - \pi_n T_t f_\lambda\| := \lim_{n \rightarrow \infty} \sup_{x \in E_n} |T_t^{(n)} \pi_n f_\lambda(x) - \pi_n T_t f_\lambda(x)| = 0,$$

where $f_\lambda(x) := e^{-\langle \lambda, x \rangle} := e^{-(\lambda_1 x_1 + \dots + \lambda_d x_d)}$ for all $\lambda \in \mathbb{N}^d$ and $x \in E$, then $X^{(n)}$ converges in $D_E[0, \infty)$ to X as $n \rightarrow \infty$.

Proof. By assumption, $\lim_{n \rightarrow \infty} \|T_t^{(n)} \pi_n f - \pi_n T_t f\| = 0$ for all $f \in D$, where $D := \{f : E \rightarrow \mathbb{R} : f(x) = \sum_{i=1}^m a_i e^{-\langle \lambda_i, x \rangle}, m \in \mathbb{N}, \lambda_i \in \mathbb{N}^d, a_i \in \mathbb{R}\}$. Let $f \in \widehat{C}(E)$ and fix $\varepsilon > 0$. Since D is dense in $\widehat{C}(E)$ by Lemma 5.3 there exists $h \in D$ such that $\|f - h\| < \varepsilon$. It follows that

$$\begin{aligned} \|T_t^{(n)} \pi_n f - \pi_n T_t f\| &\leq \|T_t^{(n)} \pi_n (f - h)\| + \|T_t^{(n)} \pi_n h - \pi_n T_t h\| + \|\pi_n T_t (h - f)\| \\ &\leq \|T_t^{(n)}\| \|f - h\| + \|T_t^{(n)} \pi_n h - \pi_n T_t h\| + \|T_t\| \|h - f\| \\ &\leq 2\varepsilon + \|T_t^{(n)} \pi_n h - \pi_n T_t h\| \rightarrow 2\varepsilon, \quad n \rightarrow \infty. \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily we conclude that $\lim_{n \rightarrow \infty} \|T_t^{(n)} \pi_n f - \pi_n T_t f\| = 0$ for all $f \in \widehat{C}(E)$. The result follows from [8, p. 172, Theorem 2.11]. \square

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